

On Reducing Linearizability to State Reachability[★]

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Abstract. Efficient implementations of atomic objects such as concurrent stacks and queues are especially susceptible to programming errors, and necessitate automatic verification. Unfortunately their correctness criteria — linearizability with respect to given ADT specifications — are hard to verify. Even on classes of implementations where the usual temporal safety properties like control-state reachability are decidable, linearizability is undecidable.

In this work we demonstrate that verifying linearizability for certain *fixed* ADT specifications is reducible to control-state reachability, despite being harder for *arbitrary* ADTs. We effectuate this reduction for several of the most popular atomic objects. This reduction yields the first decidability results for verification without bounding the number of concurrent threads. Furthermore, it enables the application of existing safety-verification tools to linearizability verification.

1 Introduction

Efficient implementations of atomic objects such as concurrent queues and stacks are difficult to get right. Their complexity arises from the conflicting design requirements of maximizing efficiency/concurrency with preserving the appearance of atomic behavior. Their correctness is captured by *observational refinement*, which assures that all behaviors of programs using these efficient implementations would also be possible were the atomic reference implementations used instead. Linearizability [11], being an equivalent property [7, 4], is the predominant proof technique: one shows that each concurrent execution has a linearization which is a valid sequential execution according to a specification, given by an abstract data type (ADT) or reference implementation.

Verifying automatically³ that all executions of a given implementation are linearizable with respect to a given ADT is an undecidable problem [3], even on the typical classes of implementations for which the usual temporal safety properties are decidable, e.g., on finite-shared-memory programs where each thread is a finite-state machine. What makes linearization harder than typical temporal safety properties like control-state reachability is the existential quantification of a valid linearization per execution.

In this work we demonstrate that verifying linearizability for certain *fixed* ADTs is reducible to control-state reachability, despite being harder for *arbitrary* ADTs. We believe that fixing the ADT parameter of the verification problem is justified, since in practice, there are few ADTs for which specialized concurrent implementations have

[★] This work is supported in part by the VECOLIB project (ANR-14-CE28-0018).

³ Without programmer annotation — see Section 6 for further discussion.

been developed. We provide a methodology for carrying out this reduction, and instantiate it on four ADTs: the atomic queue, stack, register, and mutex.

Our reduction to control-state reachability holds on any class of implementations which is closed under intersection with regular languages⁴ and which is *data independent* — informally, that implementations can perform only read and write operations on the data values passed as method arguments. From the ADT in question, our approach relies on expressing its violations as a finite union of regular languages.

In our methodology, we express the atomic object specifications using inductive rules to facilitate the incremental construction of valid executions. For instance in our atomic queue specification, one rule specifies that a dequeue operation returning empty can be inserted in any execution, so long as each preceding enqueue has a corresponding dequeue, also preceding the inserted empty-dequeue. This form of inductive rule enables a locality to the reasoning of linearizability violations.

Intuitively, first we prove that a sequential execution is invalid if and only if some subsequence could not have been produced by one of the rules. Under certain conditions this result extends to concurrent executions: an execution is not linearizable if and only if some projection of its operations cannot be linearized to a sequence produced by one of the rules. We thus correlate the finite set of inductive rules with a finite set of classes of non-linearizable concurrent executions. We then demonstrate that each of these classes of non-linearizable executions is regular, which characterizes the violations of a given ADT as a finite union of regular languages. The fact that these classes of non-linearizable executions can be encoded as regular languages is somewhat surprising since the number of data values, and thus alphabet symbols, is, a priori, unbounded. Our encoding thus relies on the aforementioned *data independence* property.

To complete the reduction to control-state reachability, we show that linearizability is equivalent to the emptiness of the language intersection between the implementation and finite union of regular violations. When the implementation is a finite-shared-memory program with finite-state threads, this reduces to the coverability problem for Petri nets, which is decidable, and EXPSpace-complete.

To summarize, our contributions are:

- a generic reduction from linearizability to control-state reachability,
- its application to the atomic queue, stack, register, and mutex ADTs,
- the methodology enabling this reduction, which can be reused on other ADTs, and
- the first decidability results for linearizability without bounding the number of concurrent threads.

Besides yielding novel decidability results, our reduction paves the way for the application of existing safety-verification tools to linearizability verification.

Section 2 outlines basic definitions. Section 3 describes a methodology for inductive definitions of data structure specifications. In Section 4 we identify conditions under which linearizability can be reduced to control-state reachability, and demonstrate that typical atomic objects satisfy these conditions. Finally, we prove decidability of linearizability for finite-shared-memory programs with finite-state threads in Section 5. Proofs to technical results appear in the appendix.

⁴ We consider languages of well-formed method call and return actions, e.g., for which each return has a matching call.

2 Linearizability

We fix a (possibly infinite) set \mathbb{D} of *data values*, and a finite set \mathbb{M} of *methods*. We consider that methods have exactly one argument, or one return value. Return values are transformed into argument values for uniformity.⁵ In order to differentiate methods taking an argument (e.g., the *Enq* method which inserts a value into a queue) from the other methods, we identify a subset $\mathbb{M}_{in} \subseteq \mathbb{M}$ of *input* methods which do take an argument. A *method event* is composed of a method $m \in \mathbb{M}$ and a data value $x \in \mathbb{D}$, and is denoted $m(x)$. We define the *concatenation* of method-event sequences $u \cdot v$ in the usual way, and ϵ denotes the empty sequence.

Definition 1. A sequential execution is a sequence of method events,

The projection $u|_D$ of a sequential execution u to a subset $D \subseteq \mathbb{D}$ of data values is obtained from u by erasing all method events with a data value not in D . The set of projections of u is denoted $\text{proj}(u)$. We write $u \setminus x$ for the projection $u|_{\mathbb{D} \setminus \{x\}}$.

Example 1. The projection $\text{Enq}(1)\text{Enq}(2)\text{Deq}(1)\text{Enq}(3)\text{Deq}(2)\text{Deq}(3) \setminus 1$ is equal to $\text{Enq}(2)\text{Enq}(3)\text{Deq}(2)\text{Deq}(3)$.

We also fix an arbitrary infinite set \mathbb{O} of operation (identifiers). A *call action* is composed of a method $m \in \mathbb{M}$, a data value $x \in \mathbb{D}$, an operation $o \in \mathbb{O}$, and is denoted $\text{call}_o m(x)$. Similarly, a *return action* is denoted $\text{ret}_o m(x)$. The operation o is used to match return actions to their call actions.

Definition 2. A (concurrent) execution e is a sequence of call and return actions which satisfy a well-formedness property: every return has a call action before it in e , using the same tuple m, x, o , and an operation o can be used only twice in e , once in a call action, and once in a return action.

Example 2. The sequence $\text{call}_{o_1} \text{Enq}(7) \cdot \text{call}_{o_2} \text{Enq}(4) \cdot \text{ret}_{o_1} \text{Enq}(7) \cdot \text{ret}_{o_2} \text{Enq}(4)$ is an execution, while $\text{call}_{o_1} \text{Enq}(7) \cdot \text{call}_{o_2} \text{Enq}(4) \cdot \text{ret}_{o_1} \text{Enq}(7) \cdot \text{ret}_{o_1} \text{Enq}(4)$ and $\text{call}_{o_1} \text{Enq}(7) \cdot \text{ret}_{o_1} \text{Enq}(7) \cdot \text{ret}_{o_2} \text{Enq}(4)$ are not.

Definition 3. An implementation \mathcal{I} is a set of (concurrent) executions.

Implementations represent libraries whose methods are called by external programs, giving rise to the following closure properties [4]. In the following, c denotes a call action, r denotes a return action, a denotes any action, and e, e' denote executions.

- Programs can call library methods at any point in time:
 $e \cdot e' \in \mathcal{I}$ implies $e \cdot c \cdot e' \in \mathcal{I}$ so long as $e \cdot c \cdot e'$ is well formed.
- Calls can be made earlier:
 $e \cdot a \cdot c \cdot e' \in \mathcal{I}$ implies $e \cdot c \cdot a \cdot e' \in \mathcal{I}$.

⁵ Method return values are guessed nondeterministically, and validated at return points. This can be handled using the assume statements of typical formal specification languages, which only admit executions satisfying a given predicate. The argument value for methods without argument or return values, or with fixed argument/return values, is ignored.

- Returns been made later:
 $e \cdot r \cdot a \cdot e' \in \mathcal{I}$ implies $e \cdot a \cdot r \cdot e' \in \mathcal{I}$.

Intuitively, these properties hold because call and return actions are not visible to the other threads which are running in parallel.

For the remainder of this work, we consider only *completed* executions, where each call action has a corresponding return action. This simplification is sound when implementation methods can always make progress in isolation [10]: formally, for any execution e with pending operations, there exists an execution e' obtained by extending e only with the return actions of the pending operations of e . Intuitively this means that methods can always return without any help from outside threads, avoiding deadlock.

We simply reasoning on executions by abstracting them into *histories*.

Definition 4. A history is a labeled partial order $(O, <, l)$ with $O \subseteq \mathbb{O}$ and $l : O \rightarrow \mathbb{M} \times \mathbb{D}$.

The order $<$ is called the *happens-before relation*, and we say that o_1 *happens before* o_2 when $o_1 < o_2$. Since histories arise from executions, their happens-before relations are *interval orders* [4]: for distinct o_1, o_2, o_3, o_4 , if $o_1 < o_2$ and $o_3 < o_4$ then either $o_1 < o_4$, or $o_3 < o_2$. Intuitively, this comes from the fact that concurrent threads share a notion of global time. $\mathbb{D}_h \subseteq \mathbb{D}$ denotes the set of data values appearing in h .

The *history of an execution* e is defined as $(O, <, l)$ where:

- O is the set of operations which appear in e ,
- $o_1 < o_2$ iff the return action of o_1 is before the call action of o_2 in e ,
- an operation o occurring in a call action $\text{call}_o m(x)$ is labeled by $m(x)$.

Example 3. The history of the execution $\text{call}_{o_1} \text{Enq}(7) \cdot \text{call}_{o_2} \text{Enq}(4) \cdot \text{ret}_{o_1} \text{Enq}(7) \cdot \text{ret}_{o_2} \text{Enq}(4)$ is $(\{o_1, o_2\}, <, l)$ with $l(o_1) = \text{Enq}(7)$, $l(o_2) = \text{Enq}(4)$, and with $<$ being the empty order relation, since o_1 and o_2 *overlap*.

Let $h = (O, <, l)$ be a history and u a sequential execution of length n . We say that h is *linearizable with respect to* u , denoted $h \sqsubseteq u$, if there is a bijection $f : O \rightarrow \{1, \dots, n\}$ s.t.

- if $o_1 < o_2$ then $f(o_1) < f(o_2)$,
- the method event at position $f(o)$ in u is $l(o)$.

Definition 5. A history h is *linearizable with respect to a set* \mathcal{S} of sequential executions, denoted $h \sqsubseteq \mathcal{S}$, if there exists $u \in \mathcal{S}$ such that $h \sqsubseteq u$.

A set of histories H is *linearizable* with respect to \mathcal{S} , denoted $H \sqsubseteq \mathcal{S}$ if $h \sqsubseteq \mathcal{S}$ for all $h \in H$. We extend these definitions to executions according to their histories.

A sequential execution u is said to be *differentiated* if, for all input methods $m \in \mathbb{M}_{in}$, and every $x \in \mathbb{D}$, there is at most one method event $m(x)$ in u . The subset of differentiated sequential executions of a set \mathcal{S} is denoted by $\mathcal{S}_\#$. The definition extends to (sets of) executions and histories. For instance, an execution is differentiated if for all input methods $m \in \mathbb{M}_{in}$ and every $x \in \mathbb{D}$, there is at most one call action $\text{call}_o m(x)$.

Example 4. $\text{call}_{o_1} \text{Enq}(7) \cdot \text{call}_{o_2} \text{Enq}(7) \cdot \text{ret}_{o_1} \text{Enq}(7) \cdot \text{ret}_{o_2} \text{Enq}(7)$ is not differentiated, as there are two call actions with the same input method (Enq) and the same data value.

A renaming r is a function from \mathbb{D} to \mathbb{D} . Given a sequential execution (resp., execution or history) u , we denote by $r(u)$ the sequential execution (resp., execution or history) obtained from u by replacing every data value x by $r(x)$.

Definition 6. *The set of sequential executions (resp., executions or histories) \mathcal{S} is data independent if:*

- for all $u \in \mathcal{S}$, there exists $u' \in \mathcal{S}_\neq$, and a renaming r such that $u = r(u')$,
- for all $u \in \mathcal{S}$ and for all renaming r , $r(u) \in \mathcal{S}$.

When checking that a data-independent implementation \mathcal{I} is linearizable with respect to a data-independent specification \mathcal{S} , it is enough to do so for differentiated executions [1]. Thus, in the remainder of the paper, we focus on characterizing linearizability for differentiated executions, rather than arbitrary ones.

Lemma 1 (Abdulla et al. [1]). *A data-independent implementation \mathcal{I} is linearizable with respect to a data-independent specification \mathcal{S} , if and only if \mathcal{I}_\neq is linearizable with respect to \mathcal{S}_\neq .*

3 Inductively-Defined Data Structures

A data structure \mathcal{S} is given syntactically as an ordered sequence of rules R_1, \dots, R_n , each of the form $u_1 \cdot u_2 \cdots u_k \in \mathcal{S} \wedge \text{Guard}(u_1, \dots, u_k) \Rightarrow \text{Expr}(u_1, \dots, u_k) \in \mathcal{S}$, where the variables u_i are interpreted over method-event sequences, and

- $\text{Guard}(u_1, \dots, u_k)$ is a conjunction of conditions on u_1, \dots, u_k with atoms
 - $u_i \in M^*$ ($M \subseteq \mathbb{M}$)
 - $\text{matched}(m, u_i)$
- $\text{Expr}(u_1, \dots, u_k)$ is an expression $E = a_1 \cdot a_2 \cdots a_l$ where
 - u_1, \dots, u_k appear in that order, exactly once, in E ,
 - each a_i is either some u_j , a method m , or a Kleene closure m^* ($m \in \mathbb{M}$),
 - a method $m \in \mathbb{M}$ appears at most once in E .

We allow k to be 0 for base rules, such as $\epsilon \in \mathcal{S}$.

A condition $u_i \in M^*$ ($M \subseteq \mathbb{M}$) is satisfied when the methods used in u_i are all in M . The predicate $\text{matched}(m, u_i)$ is satisfied when, for every method event $m(x)$ in u_i , there exists another method event in u_i with the same data value x .

Given a sequential execution $u = u_1 \cdot \dots \cdot u_k$ and an expression $E = \text{Expr}(u_1, \dots, u_k)$, we define $\llbracket E \rrbracket$ as the set of sequential executions which can be obtained from E by replacing the methods m by a method event $m(x)$ and the Kleene closures m^* by 0 or more method events $m(x)$. All method events must use the same data value $x \in \mathbb{D}$.

A rule $R \equiv u_1 \cdot u_2 \cdots u_k \in \mathcal{S} \wedge \text{Guard}(u_1, \dots, u_k) \Rightarrow \text{Expr}(u_1, \dots, u_k) \in \mathcal{S}$ is applied to a sequential execution w to obtain a new sequential execution w' from the set:

$$\bigcup_{\substack{w = w_1 \cdot w_2 \cdots w_k \wedge \\ \text{Guard}(w_1, \dots, w_k)}} \llbracket \text{Expr}(w_1, \dots, w_k) \rrbracket$$

We denote this $w \xrightarrow{R} w'$. The set of sequential executions $\llbracket S \rrbracket = \llbracket R_1, \dots, R_n \rrbracket$ is then defined as the set of sequential executions w which can be derived from the empty word:

$$\epsilon = w_0 \xrightarrow{R_{i_1}} w_1 \xrightarrow{R_{i_2}} w_2 \dots \xrightarrow{R_{i_p}} w_p = w,$$

where i_1, \dots, i_p is a non-decreasing sequence of integers from $\{1 \dots, n\}$. This means that the rules must be applied in order, and each rule can be applied 0 or several times.

Below we give inductive definitions for the atomic queue and stack data structures. Other data structures such as atomic registers and mutexes also have inductive definitions, as demonstrated in the appendix.

Example 5. The queue has a method *Enq* to add an element to the data structure, and a method *Deq* to remove the elements in a FIFO order. The method *DeqEmpty* can only return when the queue is empty (its parameter is not used). The only input method is *Enq*. Formally, Queue is defined by the rules R_0, R_{Enq}, R_{EnqDeq} and $R_{DeqEmpty}$.

$$\begin{aligned} R_0 &\equiv \epsilon \in \text{Queue} \\ R_{Enq} &\equiv u \in \text{Queue} \wedge u \in \text{Enq}^* \Rightarrow u \cdot \text{Enq} \in \text{Queue} \\ R_{EnqDeq} &\equiv u \cdot v \in \text{Queue} \wedge u \in \text{Enq}^* \wedge v \in \{\text{Enq}, \text{Deq}\}^* \Rightarrow \text{Enq} \cdot u \cdot \text{Deq} \cdot v \in \text{Queue} \\ R_{DeqEmpty} &\equiv u \cdot v \in \text{Queue} \wedge \text{matched}(\text{Enq}, u) \Rightarrow u \cdot \text{DeqEmpty} \cdot v \in \text{Queue} \end{aligned}$$

One derivation for Queue is:

$$\begin{aligned} \epsilon \in \text{Queue} &\xrightarrow{R_{EnqDeq}} \text{Enq}(1) \cdot \text{Deq}(1) \in \text{Queue} \\ &\xrightarrow{R_{EnqDeq}} \text{Enq}(2) \cdot \text{Enq}(1) \cdot \text{Deq}(2) \cdot \text{Deq}(1) \in \text{Queue} \\ &\xrightarrow{R_{EnqDeq}} \text{Enq}(3) \cdot \text{Deq}(3) \cdot \text{Enq}(2) \cdot \text{Enq}(1) \cdot \text{Deq}(2) \cdot \text{Deq}(1) \in \text{Queue} \\ &\xrightarrow{R_{DeqEmpty}} \text{Enq}(3) \cdot \text{Deq}(3) \cdot \text{DeqEmpty} \cdot \text{Enq}(2) \cdot \text{Enq}(1) \cdot \text{Deq}(2) \cdot \text{Deq}(1) \in \text{Queue} \end{aligned}$$

Similarly, Stack is composed of the rules $R_0, R_{PushPop}, R_{Push}, R_{PopEmpty}$.

$$\begin{aligned} R_0 &\equiv \epsilon \in \text{Stack} \\ R_{PushPop} &\equiv u \cdot v \in \text{Stack} \wedge \text{matched}(\text{Push}, u) \wedge \text{matched}(\text{Push}, v) \wedge u, v \in \{\text{Push}, \text{Pop}\}^* \\ &\Rightarrow \text{Push} \cdot u \cdot \text{Pop} \cdot v \in \text{Stack} \\ R_{Push} &\equiv u \cdot v \in \text{Stack} \wedge \text{matched}(\text{Push}, u) \wedge u, v \in \{\text{Push}, \text{Pop}\}^* \Rightarrow u \cdot \text{Push} \cdot v \in \text{Stack} \\ R_{PopEmpty} &\equiv u \cdot v \in \text{Stack} \wedge \text{matched}(\text{Push}, u) \Rightarrow u \cdot \text{PopEmpty} \cdot v \in \text{Stack} \end{aligned}$$

We assume that the rules defining a data structure S satisfy a non-ambiguity property stating that the last step in deriving a sequential execution in $\llbracket S \rrbracket$ is unique and it can be effectively determined. Since we are interested in characterizing the linearizations of a history and its projections, this property is extended to permutations of projections of sequential executions which are admitted by S . Thus, we assume that the rules defining a data structure are *non-ambiguous*, that is:

- for all $u \in \llbracket S \rrbracket$, there exists a unique rule, denoted by $\text{last}(u)$, that can be used as the last step to derive u , i.e., for every sequence of rules R_{i_1}, \dots, R_{i_n} leading to u , $R_{i_n} = \text{last}(u)$. For $u \notin \llbracket S \rrbracket$, $\text{last}(u)$ is also defined but can be arbitrary, as there is no derivation for u .

- if $\text{last}(u) = R_i$, then for every permutation $u' \in \llbracket S \rrbracket$ of a projection of u , $\text{last}(u') = R_j$ with $j \leq i$. If u' is a permutation of u , then $\text{last}(u') = R_i$.

Given a (completed) history h , all the u such that $h \sqsubseteq u$ are permutations of one another. The last condition of non-ambiguity thus enables us to extend the function last to histories: $\text{last}(h)$ is defined as $\text{last}(u)$ where u is any sequential execution such that $h \sqsubseteq u$. We say that $\text{last}(h)$ is the rule *corresponding* to h .

Example 6. For Queue, we define last for a sequential execution u as follows:

- if u contains a *DeqEmpty* operation, $\text{last}(u) = R_{\text{DeqEmpty}}$,
- else if u contains a *Deq* operation, $\text{last}(u) = R_{\text{EnqDeq}}$,
- else if u contains only *Enq*'s, $\text{last}(u) = R_{\text{Enq}}$,
- else (if u is empty), $\text{last}(u) = R_0$.

Since the conditions we use to define last are closed under permutations, we get that for any permutation u_2 of u , $\text{last}(u) = \text{last}(u_2)$, and last can be extended to histories. Therefore, the rules $R_0, R_{\text{EnqDeq}}, R_{\text{DeqEmpty}}$ are non-ambiguous.

4 Reducing Linearizability to State Reachability

Our end goal for this section is to show that for any data-independent implementation I , and any specification S satisfying several conditions defined in the following, there exists a computable finite-state automaton \mathcal{A} (over call and return actions) such that:

$$I \sqsubseteq S \iff I \cap \mathcal{A} = \emptyset$$

Then, given a model of I , the linearizability of I is reduced to checking emptiness of the synchronized product between the model of I and \mathcal{A} . The automaton \mathcal{A} represents (a subset of the) executions which are not linearizable with respect to S .

The first step in proving our result is to show that, under some conditions, we can partition the concurrent executions which are not linearizable with respect to S into a finite number of classes. Intuitively, each non-linearizable execution must correspond to a violation for one of the rules in the definition of S .

We identify a property, which we call *step-by-step linearizability*, which is sufficient to obtain this characterization. Intuitively, step-by-step linearizability enables us to build a linearization for an execution e incrementally, using linearizations of projections of e .

The second step is to show that, for each class of violations (i.e., with respect to a specific rule R_i), we can build a regular automaton \mathcal{A}_i such that: a) when restricted to well-formed executions, \mathcal{A}_i recognizes a subset of this class; b) each non-linearizable execution has a corresponding execution, obtained by data independence, accepted by \mathcal{A}_i . If such an automaton exists, we say that R_i is *co-regular* (formally defined later in this section).

We prove that, provided these two properties hold, we have the equivalence mentioned above, by defining \mathcal{A} as the union of the \mathcal{A}_i 's built for each rule R_i .

4.1 Reduction to a Finite Number of Classes of Violations

Our goal here is to give a characterization of the sequential executions which belong to a data structure, as well as to give a characterization of the concurrent executions which are linearizable with respect to the data structure. This characterization enables us to classify the linearization violations into a finite number of classes.

Our characterization relies heavily on the fact that the data structures we consider are *closed under projection*, i.e., for all $u \in \mathcal{S}$, $D \subseteq \mathbb{D}$, we have $u|_D \in \mathcal{S}$. The reason for this is that the guards used in the inductive rules are closed under projection.

Lemma 2. *Any data structure \mathcal{S} defined in our framework is closed under projection.*

A sequential execution u is said to *match* a rule R with conditions *Guard* if there exist a data value x and sequential executions u_1, \dots, u_k such that u can be written as $\llbracket \text{Expr}(u_1, \dots, u_k) \rrbracket$, where x is the data value used for the method events, and such that $\text{Guard}(u_1, \dots, u_k)$ holds. We call x the *witness* of the decomposition. We denote by MR the set of sequential executions which match R , and we call it the *matching set* of R .

Example 7. MR_{EnqDeq} is the set of sequential executions of the form $\text{Enq}(x) \cdot u \cdot \text{Deq}(x) \cdot v$ for some $x \in \mathbb{D}$, and with $u \in \text{Enq}^*$.

Lemma 3. *Let $\mathcal{S} = R_1, \dots, R_n$ be a data structure and u a differentiated sequential execution. Then,*

$$u \in \mathcal{S} \iff \text{proj}(u) \subseteq \bigcup_{i \in \{1, \dots, n\}} MR_i$$

This characterization enables us to get rid of the recursion, so that we only have to check non-recursive properties. We want a similar lemma to characterize $e \sqsubseteq \mathcal{S}$ for an execution e . This is where we introduce the notion of *step-by-step linearizability*, as the lemma will hold under this condition.

Definition 7. *A data structure $\mathcal{S} = R_1, \dots, R_n$ is said to be step-by-step linearizable if for any differentiated execution e , if e is linearizable w.r.t. MR_i with witness x , we have:*

$$e \setminus x \sqsubseteq \llbracket R_1, \dots, R_i \rrbracket \implies e \sqsubseteq \llbracket R_1, \dots, R_i \rrbracket$$

This notion applies to the usual data structures, as shown by the following lemma. The generic schema we use is the following: we let $u' \in \llbracket R_1, \dots, R_i \rrbracket$ be a sequential execution such that $e \setminus x \sqsubseteq u'$ and build a graph G from u' , whose acyclicity implies that $e \sqsubseteq \llbracket R_1, \dots, R_i \rrbracket$. Then, we show that we can always choose u' so that G is acyclic.

Lemma 4. *Queue, Stack, Register, and Mutex are step-by-step linearizable.*

Intuitively, step-by-step linearizability will help us prove the right-to-left direction of Lemma 5 by allowing us to build a linearization for e incrementally, from the linearizations of projections of e .

Lemma 5. *Let \mathcal{S} be a data structure with rules R_1, \dots, R_n . Let e be a differentiated execution. If \mathcal{S} is step-by-step linearizable, we have (for any j):*

$$e \sqsubseteq \llbracket R_1, \dots, R_j \rrbracket \iff \text{proj}(e) \sqsubseteq \bigcup_{i \leq j} MR_i$$

Thanks to Lemma 5, if we're looking for an execution e which is not linearizable w.r.t. some data-structure \mathcal{S} , we must prove that $\text{proj}(e) \not\sqsubseteq \bigcup_i MR_i$, i.e., we must find a projection $e' \in \text{proj}(e)$ which is not linearizable with respect to any MR_i ($e' \not\sqsubseteq \bigcup_i MR_i$).

This is challenging as it is difficult to check that an execution is not linearizable w.r.t. a union of sets simultaneously. Using non-ambiguity, we simplify this check by making it more modular, so that we only have to check one set MR_i at a time.

Lemma 6. *Let \mathcal{S} be a data structure with rules R_1, \dots, R_n . Let e be a differentiated execution. If \mathcal{S} is step-by-step linearizable, we have:*

$$e \sqsubseteq \mathcal{S} \iff \forall e' \in \text{proj}(e). e' \sqsubseteq MR \text{ where } R = \text{last}(e')$$

Lemma 6 gives us the finite kind of violations that we mentioned in the beginning of the section. More precisely, if we negate both sides of the equivalence, we have: $e \not\sqsubseteq \mathcal{S} \iff \exists e' \in \text{proj}(e). e' \not\sqsubseteq MR$. This means that whenever an execution is not linearizable w.r.t. \mathcal{S} , there can be only finitely reasons, namely there must exist a projection which is not linearizable w.r.t. the matching set of its corresponding rule.

4.2 Regularity of Each Class of Violations

Our goal is now to construct, for each R , an automaton \mathcal{A} which recognizes (a subset of) the executions e , which have a projection e' such that $e' \not\sqsubseteq MR$. More precisely, we want the following property.

Definition 8. *A rule R is said to be co-regular if we can build an automaton \mathcal{A} such that, for any data-independent implementation \mathcal{I} , we have:*

$$\mathcal{A} \cap \mathcal{I} \neq \emptyset \iff \exists e \in \mathcal{I}_\#, e' \in \text{proj}(e). \text{last}(e') = R \wedge e' \not\sqsubseteq MR$$

A data structure \mathcal{S} is co-regular if all of its rules are co-regular.

Formally, the alphabet of \mathcal{A} is $\{\text{call } m(x) \mid m \in \mathbb{M}, x \in D\} \cup \{\text{ret } m(x) \mid m \in \mathbb{M}, x \in D\}$ for a finite subset $D \subseteq \mathbb{D}$. The automaton doesn't read operation identifiers, thus, when taking the intersection with \mathcal{I} , we ignore them.

Lemma 7. *Queue, Stack, Register, and Mutex are co-regular.*

Proof. To illustrate this lemma, we sketch the proof for the rule $R_{DeqEmpty}$ of Queue. The complete proof of the lemma can be found in the extended version of this paper.

We prove in the appendix (Corollary 1) that a history has a projection such that $\text{last}(h') = R_{DeqEmpty}$ and $h' \not\sqsubseteq MR_{DeqEmpty}$ if and only if it has a *DeqEmpty* operation which is *covered* by other operations, as depicted in Fig. 1. The automaton $\mathcal{A}_{R_{DeqEmpty}}$ in Fig. 2 recognizes such violations.

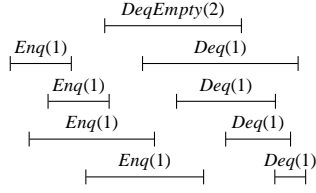


Fig. 1. A four-pair $R_{DeqEmpty}$ violation. Lemma 19 demonstrates that this pattern with arbitrarily-many pairs is regular.

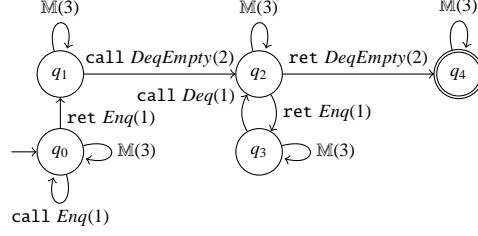


Fig. 2. An automaton recognizing $R_{DeqEmpty}$ violations, for which the queue is non-empty, with data value 1, for the span of $DeqEmpty$. We assume all call $Enq(1)$ actions occur initially without loss of generality due to implementations' closure properties.

Let \mathcal{I} be any data-independent implementation. We show that

$$\mathcal{A}_{R_{DeqEmpty}} \cap \mathcal{I} \neq \emptyset \iff \exists e \in \mathcal{I}_\neq, e' \in \text{proj}(e). \text{last}(e') = R_{DeqEmpty} \wedge e' \not\sqsubseteq MR_{DeqEmpty}$$

(\Rightarrow) Let $e \in \mathcal{I}$ be an execution which is accepted by $\mathcal{A}_{R_{DeqEmpty}}$. By data independence, let $e_\neq \in \mathcal{I}$ and r a renaming such that $e = r(e_\neq)$. Let d_1, \dots, d_m be the data values which are mapped to value 1 by r .

Let d be the data value which is mapped to value 2 by r . Let o the $DeqEmpty$ operation with data value d . By construction of the automaton we can prove that o is covered by d_1, \dots, d_m , and using Corollary 1, conclude that h has a projection such that $\text{last}(h') = R_{DeqEmpty}$ and $h' \not\sqsubseteq MR_{DeqEmpty}$.

(\Leftarrow) Let $e_\neq \in \mathcal{I}_\neq$ such that there is a projection e' such that $\text{last}(e') = R_{DeqEmpty}$ and $e' \not\sqsubseteq MR_{DeqEmpty}$. Let d_1, \dots, d_m be the data values given by Corollary 1, and let d be the data value corresponding to the $DeqEmpty$ operation.

Without loss of generality, we can always choose the cycle so that $Enq(d_i)$ doesn't happen before $Deq(d_{i-2})$ (if it does, drop d_{i-1}).

Let r be the renaming which maps d_1, \dots, d_m to 1, d to 2, and all other values to 3. Let $e = r(e_\neq)$. The execution e can be recognized by automaton $\mathcal{A}_{R_{DeqEmpty}}$, and belongs to \mathcal{I} by data independence.

When we have a data structure which is both step-by-step linearizable and co-regular, we can make a linear time reduction from the verification of linearizability with respect to \mathcal{S} to a reachability problem, as illustrated in Theorem 1.

Theorem 1. *Let \mathcal{S} be a step-by-step linearizable and co-regular data structure and let \mathcal{I} be a data-independent implementation. There exists a regular automaton \mathcal{A} such that:*

$$\mathcal{I} \sqsubseteq \mathcal{S} \iff \mathcal{I} \cap \mathcal{A} = \emptyset$$

5 Decidability and Complexity of Linearizability

Theorem 1 implies that the linearizability problem with respect to any step-by-step linearizable and co-regular specification is decidable for any data-independent implemen-

tation for which checking the emptiness of the intersection with finite-state automata is decidable. Here, we give a class C of data-independent implementations for which the latter problem, and thus linearizability, is decidable.

Each method of an implementation in C manipulates a finite number of local variables which store Boolean values, or data values from \mathbb{D} . Methods communicate through a finite number of shared variables that also store Boolean values, or data values from \mathbb{D} . Data values may be assigned, but never used in program predicates (e.g., in the conditions of `if` and `while` statements) so as to ensure data independence. This class captures typical implementations, or finite-state abstractions thereof, e.g., obtained via predicate abstraction.

Let \mathcal{I} be an implementation from class C . The automata \mathcal{A} constructed in the proof of Lemma 7 use only data values 1, 2, and 3. Checking emptiness of $\mathcal{I} \cap \mathcal{A}$ is thus equivalent to checking emptiness of $\mathcal{I}_3 \cap \mathcal{A}$ with the three-valued implementation $\mathcal{I}_3 = \{e \in \mathcal{I} \mid e = e_{\{1,2,3\}}\}$. The set \mathcal{I}_3 can be represented by a Petri net since bounding data values allows us to represent each thread with a finite-state machine. Intuitively, each token in the Petri net represents another thread. The number of threads can be unbounded since the number of tokens can. Places count the number of threads in each control location, which includes a local-variable valuation. Each shared variable also has one place per value to store its current valuation.

Emptiness of the intersection with regular automata reduces to the EXPSPACE-complete coverability problem for Petri nets. Limiting verification to a bounded number of threads lowers the complexity of coverability to PSPACE [6]. The hardness part of Theorem 2 comes from the hardness of state reachability in finite-state concurrent programs.

Theorem 2. *Verifying linearizability of an implementation in C with respect to a step-by-step linearizable and co-regular specification is PSPACE-complete for a fixed number of threads, and EXPSPACE-complete otherwise.*

6 Related Work

Several works investigate the theoretical limits of linearizability verification. Verifying a single execution against an arbitrary ADT specification is NP-complete [8]. Verifying all executions of a finite-state implementation against an arbitrary ADT specification (given as a regular language) is EXPSPACE-complete when program threads are bounded [2, 9], and undecidable otherwise [3].

Existing automated methods for proving linearizability of an atomic object implementation are also based on reductions to safety verification [1, 10, 12]. Vafeiadis [12] considers implementations where operation’s *linearization points* are fixed to particular source-code locations. Essentially, this approach instruments the implementation with ghost variables simulating the ADT specification at linearization points. This approach is incomplete since not all implementations have fixed linearization points. Aspect-oriented proofs [10] reduce linearizability to the verification of four simpler safety properties. However, this approach has only been applied to queues, and has not produced a fully automated and complete proof technique. Dodds et al. [5] prove linearizability of

stack implementations with an automated proof assistant. Their approach does not lead to full automation however, e.g., by reduction to safety verification.

7 Conclusion

We have demonstrated a linear-time reduction from linearizability for fixed ADT specifications to control-state reachability, and the application of this reduction to atomic queues, stacks, registers, and mutexes. Besides yielding novel decidability results, our reduction enables the use of existing safety-verification tools for linearizability. While this work only applies the reduction to these four objects, our methodology also applies to other typical atomic objects including semaphores and sets. Although this methodology currently does not capture priority queues, which are not data independent, we believe our approach can be extended to include them. We leave this for future work.

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8 Appendix

8.1 Examples

For all examples, the domain \mathbb{D} is the set of natural numbers \mathbb{N} .

Stack Definition of the function last for a sequential execution u :

- if u contains a *PopEmpty* operation, $\text{last}(u) = R_{\text{PopEmpty}}$,
- else if u contains an unmatched *Push* operation, $\text{last}(u) = R_{\text{Push}}$,
- else if u contains a *Pop* operation, $\text{last}(u) = R_{\text{PushPop}}$,
- else (if u is empty), $\text{last}(u) = R_0$.

Register The register has a method *Write* used to write a data-value, and a method *Read* which returns the last written value. The only input method is *Write*. Its rules are R_0 and R_{WR} :

$$\begin{aligned} R_0 &\equiv \epsilon \in \text{Register} \\ R_{WR} &\equiv u \in \text{Register} \Rightarrow \text{Write} \cdot \text{Read}^* \cdot u \in \text{Register} \end{aligned}$$

Definition of the function last for a sequential execution u :

- if u is not empty, $\text{last}(u) = R_{WR}$,
- else, $\text{last}(u) = R_0$.

Mutex (Lock) The mutex has a method *Lock*, used to take ownership of the Mutex, and a method *Unlock*, to release it. The only input method is *Lock*. It is composed of the rules R_0 , R_{Lock} and R_{LU} :

$$\begin{aligned} R_0 &\equiv \epsilon \in \text{Mutex} \\ R_{Lock} &\equiv \text{Lock} \in \text{Mutex} \\ R_{LU} &\equiv u \in \text{Mutex} \Rightarrow \text{Lock} \cdot \text{Unlock} \cdot u \in \text{Mutex} \end{aligned}$$

In practice, *Lock* and *Unlock* methods do not have a parameter. Here, the parameter represents a *ghost variable* which helps us relate *Unlock* to their corresponding *Lock*. Any implementation will be data independent with respect to these ghost variables.

Definition of the function last for a sequential execution u :

- if u contains an *Unlock* operation, $\text{last}(u) = R_{LU}$,
- else if u is not empty, $\text{last}(u) = R_{Lock}$,
- else, $\text{last}(u) = R_0$.

8.2 Proofs of Section 4

Lemma 1 (Abdulla et al. [1]). *A data-independent implementation \mathcal{I} is linearizable with respect to a data-independent specification \mathcal{S} , if and only if $\mathcal{I}_\#$ is linearizable with respect to $\mathcal{S}_\#$.*

Proof. (\Rightarrow) Let e be a (differentiated) execution in $\mathcal{I}_\#$. By assumption, it is linearizable with respect to a sequential execution u in \mathcal{S} , and the bijection between the operations of e and the method events of u , ensures that u is differentiated and belongs to $\mathcal{S}_\#$.

(\Leftarrow) Let e be an execution in \mathcal{I} . By data independence of \mathcal{I} , we know there exists $e_\# \in \mathcal{I}_\#$ and a renaming r such that $r(e_\#) = e$. By assumption, $e_\#$ is linearizable with respect to a sequential execution $u_\# \in \mathcal{S}_\#$. We define $u = r(u_\#)$, and know by data independence of \mathcal{S} that $u \in \mathcal{S}$. Moreover, we can use the same bijection used for $e_\# \sqsubseteq u_\#$ to prove that $e \sqsubseteq u$.

Lemma 2. *Any data structure \mathcal{S} defined in our framework is closed under projection.*

Proof. Let $u \in \mathcal{S}$ and let $D \subseteq \mathbb{D}$. Since $u \in \mathcal{S}$, there is a sequence of applications of rules starting from the empty word ϵ which can derive u . We remove from this derivation all the rules corresponding to a data-value $x \notin D$, and we project all the sequential executions appearing in the derivation on the D . Since the predicates which appear in the conditions are all closed under projection, the derivation remains valid, and proves that $u|_D \in \mathcal{S}$.

Lemma 3. *Let $\mathcal{S} = R_1, \dots, R_n$ be a data structure and u a differentiated sequential execution. Then,*

$$u \in \mathcal{S} \iff \text{proj}(u) \subseteq \bigcup_{i \in \{1, \dots, n\}} MR_i$$

Proof. (\Rightarrow) Using Lemma 2, we know that \mathcal{S} is closed under projection. Thus, any projection of a sequential execution u of \mathcal{S} is itself in \mathcal{S} and has to match one of the rules R_1, \dots, R_n .

(\Leftarrow) By induction on the size of u . We know $u \in \text{proj}(u)$, so it can be decomposed to satisfy the conditions *Guard* of some rule R of \mathcal{S} . The recursive condition is then verified by induction.

Lemma 5. *Let \mathcal{S} be a data structure with rules R_1, \dots, R_n . Let e be a differentiated execution. If \mathcal{S} is step-by-step linearizable, we have (for any j):*

$$e \sqsubseteq \llbracket R_1, \dots, R_j \rrbracket \iff \text{proj}(e) \subseteq \bigcup_{i \leq j} MR_i$$

Proof. (\Rightarrow) We know there exists $u \in \mathcal{S}$ such that $e \sqsubseteq u$. Each projection e' of e can be linearized with respect to some projection u' of u , which belongs to $\bigcup_i MR_i$ according to Lemma 3.

(\Leftarrow) By induction on the size of e . We know $e \in \text{proj}(e)$ so it can be linearized with respect to a sequential execution u matching some rule R_k ($k < j$) with some witness x . Let $e' = e \setminus x$.

Since \mathcal{S} is non-ambiguous, we know that no projection of e can be linearized to a matching set MR_i with $i > k$, and in particular no projection of e' . Thus, we deduce that $\text{proj}(e') \subseteq \bigcup_{i \leq k} MR_i$, and conclude by induction that $e' \subseteq \llbracket R_1, \dots, R_k \rrbracket$.

We finally use the fact that \mathcal{S} is step-by-step linearizable to deduce that $e \subseteq \llbracket R_1, \dots, R_k \rrbracket$ and $e \subseteq \llbracket R_1, \dots, R_j \rrbracket$ because $k < j$.

Lemma 6. *Let \mathcal{S} be a data structure with rules R_1, \dots, R_n . Let e be a differentiated execution. If \mathcal{S} is step-by-step linearizable, we have:*

$$e \subseteq \mathcal{S} \iff \forall e' \in \text{proj}(e). e' \subseteq MR \text{ where } R = \text{last}(e')$$

Proof. (\Rightarrow) Let $e' \in \text{proj}(e)$. By Lemma 5, we know that e' is linearizable with respect to MR_i for some i . Since \mathcal{S} is non-ambiguous, $\text{last}(e')$ is the only rule such that $e' \subseteq MR$ can hold, which ends this part of the proof.

(\Leftarrow) Particular case of Lemma 5.

Theorem 1. *Let \mathcal{S} be a step-by-step linearizable and co-regular data structure and let \mathcal{I} be a data-independent implementation. There exists a regular automaton \mathcal{A} such that:*

$$\mathcal{I} \subseteq \mathcal{S} \iff \mathcal{I} \cap \mathcal{A} = \emptyset$$

Proof. Let $\mathcal{A}_1, \dots, \mathcal{A}_n$ be the regular automata used to show that R_1, \dots, R_n are co-regular, and let \mathcal{A} be the (non-deterministic) union of the \mathcal{A}_i 's.

(\Rightarrow) Assume there exists an execution $e \in \mathcal{I} \cap \mathcal{A}$. For some i , $e \in \mathcal{A}_i$. From the definition of “co-regular”, we deduce that there exists $e' \in \text{proj}(e)$ such that $e' \not\subseteq MR_i$, where R_i is the rule corresponding to e' . By Lemma 6, e is not linearizable with respect to \mathcal{S} .

(\Leftarrow) Assume there exists an execution $e \in \mathcal{I}$ which is not linearizable with respect to \mathcal{S} . By Lemma 6, it has a projection $e' \in \text{proj}(e)$ such that $e' \not\subseteq MR_i$, where R_i is the rule corresponding to e' . By definition of “co-regular”, this means that $\mathcal{I} \cap \mathcal{A}_i \neq \emptyset$, and that $\mathcal{I} \cap \mathcal{A} \neq \emptyset$.

8.3 Step-by-step Linearizability

Lemma 4. *Queue, Stack, Register, and Mutex are step-by-step linearizable.*

Proof. Even though we do not have a unique proof that the data structures are step-by-step linearizable, we have a model of proof which is generic, which we use for each data structure. The generic schema we use is the following: we let $u' \in \llbracket R_1, \dots, R_i \rrbracket$ be a sequential execution such that $h \setminus x \subseteq u'$ and build a graph G from u' , whose acyclicity implies that $h \subseteq \llbracket R_1, \dots, R_i \rrbracket$. Then we show that we can always choose u' so that this G is acyclic.

For better readability we make a sublemma per data structure.

Lemma 8. *Queue is step-by-step linearizable.*

Proof. Let h be a differentiated history, and u a sequential execution such that $h \sqsubseteq u$. We have three cases to consider:

1) u matches R_{Enq} with witness x : let $h' = h \setminus x$ and assume $h' \sqsubseteq \llbracket R_0, R_{Enq} \rrbracket$. Since u matches R_{Enq} , we know h only contain Enq operations. The set $\llbracket R_0, R_{Enq} \rrbracket$ is composed of the sequential executions formed by repeating the Enq method events, which means that $h \sqsubseteq \llbracket R_0, R_{Enq} \rrbracket$.

2) u matches R_{EnqDeq} with witness x : let $h' = h \setminus x$ and assume $h' \sqsubseteq \llbracket R_0, R_{Enq}, R_{EnqDeq} \rrbracket$. Let $u' \in \llbracket R_0, R_{Enq}, R_{EnqDeq} \rrbracket$ such that $h' \sqsubseteq u'$. We define a graph G whose nodes are the operations of h and there is an edge from operation o_1 to o_2 if

1. o_1 happens-before o_2 in h ,
2. the method event corresponding to o_1 in u' is before the one corresponding to o_2 ,
3. $o_1 = Enq(x)$ and o_2 is any other operation,
4. $o_1 = Deq(x)$ and o_2 is any other Deq operation.

If G is acyclic, any total order compatible with G forms a sequence u_2 such that $h \sqsubseteq u_2$ and such that u_2 can be built from u' by adding $Enq(x)$ at the beginning and $Deq(x)$ before all Deq method events. Thus, $u_2 \in \llbracket R_0, R_{Enq}, R_{EnqDeq} \rrbracket$ and $h \sqsubseteq \llbracket R_0, R_{Enq}, R_{EnqDeq} \rrbracket$.

Assume that G has a cycle, and consider a cycle C of minimal size. We show that there is only one kind of cycle possible, and that this cycle can be avoided by choosing u' appropriately. Such a cycle can only contain one happens-before edge (edges of type 1), because if there were two, we could apply the interval order property to reduce the cycle. Similarly, since the order imposed by u' is a total order, it also satisfies the interval order property, meaning that C can only contain one edge of type 2.

Moreover, C can also contain only one edge of type 3, otherwise it would have to go through $Enq(x)$ more than once. Similarly, it can contain only one edge of type 4. It cannot contain a type 3 edge $Enq(x) \rightarrow o_1$ at the same time as a type 4 edge $Deq(x) \rightarrow o_2$, because we could shortcut the cycle by a type 3 edge $Enq(x) \rightarrow o_2$.

Finally, it cannot be a cycle of size 2. For instance, a type 2 edge cannot form a cycle with a type 1 edge because $h' \sqsubseteq u'$. The only form of cycles left are the two cycles of size 3 where:

- $Enq(x)$ is before o_1 (type 3), o_1 is before o_2 in u' (type 2), and o_2 happens-before $Enq(x)$: this is not possible, because h is linearizable with respect to u which matches R_{EnqDeq} with x as a witness. This means that u starts with the method event $Enq(x)$, and that no operation can happen-before $Enq(x)$ in h .
- $Deq(x)$ is before o_1 (type 4), o_1 is before o_2 in u' (type 2), and o_2 happens-before $Deq(x)$: by definition, we know that o_1 is a Deq operation; moreover, since h is linearizable with respect to u which matches R_{EnqDeq} with x as a witness, no Deq operation can happen-before $Deq(x)$ in h , and o_2 is an Enq operation (or Enq). Let $d_1, d_2 \in \mathbb{D}$ such that $Deq(d_1) = o_1$ and $Enq(d_2) = o_2$.

Since o_1 is before o_2 in u' , we know that d_1 and d_2 must be different. Moreover, there is no happens-before edge from o_1 to o_2 , or otherwise, by transitivity of the happens-before relation, we'd have a cycle of size 2 between o_1 and $Deq(x)$.

Assume without loss of generality that o_1 is the rightmost Deq method event which is before o_2 in u' , and let o_2^1, \dots, o_2^s be the Enq (or Enq) method events between o_1 and o_2 . There is no happens-before edge $o_1 \leq_{hb} o_2^i$, because by applying the interval

order property with the other happens-before edge $o_2 \leq_{hb} Deq(x)$, we'd either have $o_1 \leq_{hb} Deq(x)$ (forming a cycle of size 2) or $o_2 \leq_{hb} o_2^i$ (not possible because $h' \sqsubseteq u'$ and o_2^i is before o_2 in u').

Let u'_2 be the sequence u' where $Deq(x)$ has been moved after o_2 . Since we know there is no happens-before edge from $Deq(x)$ to o_2^i or to o_2 , we can deduce that: $h' \sqsubseteq u'_2$. Moreover, if we consider the sequence of deductions which proves that $u' \in \llbracket R_0, R_{Enq}, R_{EnqDeq} \rrbracket$, we can alter it when we insert the pair $Enq(d_1)$ and $o_1 = Deq(d_1)$ by inserting o_1 after the o_2^i 's and after o_2 , instead of before (the conditions of the rule R_{EnqDeq} allow it).

This concludes case 2), as we're able to choose u' so that G is acyclic, and prove that $h \sqsubseteq \llbracket R_0, R_{Enq}, R_{EnqDeq} \rrbracket$.

3) u matches $R_{DeqEmpty}$ with witness x : let o be the $DeqEmpty$ operation corresponding to the witness. Let $h' = h \setminus x$ and assume $h' \sqsubseteq Queue$. Let L be the set of operations which are before o in u , and R the ones which are after. Let D_L be the data-values appearing in L and D_R be the data-values appearing in R . Since u matches $R_{DeqEmpty}$, we know that L contains no unmatched Enq operations.

Let $u' \in Queue$ such that $h' \sqsubseteq u'$. Let $u'_L = u'|_{D_L}$ and $u'_R = u'|_{D_R}$. Since $Queue$ is closed under projection, $u'_L, u'_R \in Queue$. Let $u_2 = u'_L \cdot o \cdot u'_R$. We can show that $u_2 \in Queue$ by using the derivations of u'_L and u'_R . Intuitively, this is because $Queue$ is closed under concatenation when the left-hand sequential execution has no unmatched Enq method event, like u'_L .

Moreover, we have $h \sqsubseteq u_2$, as shown in the following. We define a graph G whose nodes are the operations of h and there is an edge from operation o_1 to o_2 if

1. o_1 happens-before o_2 in h ,
2. the method event corresponding to o_1 in u_2 is before the one corresponding to o_2 .

Assume there is a cycle in G , meaning there exists o_1, o_2 such that o_1 happens-before o_2 in h , but the corresponding method events are in the opposite order in u_2 .

- If $o_1, o_2 \in L$, or $o_1, o_2 \in R$, this contradicts $h' \sqsubseteq u'$.
- If $o_1 \in R$ and $o_2 \in L$, this contradicts $h \sqsubseteq u$.
- If $o_1 \in R$ and $o_2 = o$, or if $o_1 = o$ and $o_2 \in L$, this contradicts $h \sqsubseteq u$.

This shows that $h \sqsubseteq u_2$. Thus, we have $h \sqsubseteq Queue$ and concludes the proof that the $Queue$ is step-by-step linearizable.

Lemma 9. *Stack is step-by-step linearizable.*

Proof. Let h be a differentiated history, and u a sequential execution such that $h \sqsubseteq u$. We have three cases to consider:

1) (very similar to case 3 of the $Queue$) u matches $R_{PushPop}$ with witness x : let a and b be respectively the Push and Pop operations corresponding to the witness. Let $h' = h \setminus x$ and assume $h' \sqsubseteq \llbracket R_{PushPop} \rrbracket$. Let L be the set of operations which are before b in u , and R the ones which are after. Let D_L be the data-values appearing in L and D_R be the data-values appearing in R . Since u matches $R_{PushPop}$, we know that L contains no unmatched Push operations.

Let $u' \in \llbracket R_{PushPop} \rrbracket$ such that $h' \sqsubseteq u'$. Let $u'_L = u'|_{D_L}$ and $u'_R = u'|_{D_R}$. Since $\llbracket R_{PushPop} \rrbracket$ is closed under projection, $u'_L, u'_R \in \llbracket R_{PushPop} \rrbracket$. Let $u_2 = a \cdot u'_L \cdot b \cdot u'_R$. We can show that $u_2 \in \llbracket R_{PushPop} \rrbracket$ by using the derivations of u'_L and u'_R .

Moreover, we have $h \sqsubseteq u_2$, because if the total order of u_2 didn't respect the happens-before relation of u_2 , it could only be because of four reasons, all leading to a contradiction:

- the violation is between two L operations or two R operations, contradicting $h' \sqsubseteq u'$
- the violation is between a L and an R operation, contradicting $h \sqsubseteq u$
- the violation is between b and another operation, contradicting $h \sqsubseteq u$
- the violation is between a and another operation contradicting $h \sqsubseteq u$

This shows that $h \sqsubseteq \llbracket R_{PushPop} \rrbracket$ and concludes case 1.

2) u matches R_{Push} with witness x : similar to case 1

3) u matches $R_{PopEmpty}$ with witness x : identical to case 3 of the Queue

Lemma 10. *Register is step-by-step linearizable.*

Proof. Let h be a differentiated history, and u a sequential execution such that $h \sqsubseteq u$ and such that u matches the rule R_{WR} with witness x . Let a and b_1, \dots, b_s be respectively the Write and Read's operations of h corresponding to the witness.

Let $h' = h \setminus x$ and assume $h' \sqsubseteq \llbracket R_{WR} \rrbracket$. Let $u' \in \llbracket R_{WR} \rrbracket$ such that $h' \sqsubseteq u'$. Let $u_2 = a \cdot b_1 \cdot b_2 \cdots b_s \cdot u'$. By using rule R_{WR} on u' , we have $u_2 \in \llbracket R_{WR} \rrbracket$. Moreover, we prove that $h \sqsubseteq u_2$ by contradiction. Assume that the total order imposed by u_2 doesn't respect the happens-before relation of h . All three cases are not possible:

- the violation is between two u' operations, contradicting $h' \sqsubseteq u'$,
- the violation is between a and another operation, i.e., there is an operation o which happens-before a in h , contradicting $h \sqsubseteq u$,
- the violation is between some b_i and a u' operation, i.e., there is an operation o which happens'before b_i in h , contradicting $h \sqsubseteq u$.

Thus, we have $h \sqsubseteq u_2$ and $h \sqsubseteq \llbracket R_{WR} \rrbracket$, which ends the proof.

Lemma 11. *Mutex is step-by-step linearizable.*

Proof. Identical to the Register proof, except there is only one Unlock operation (b), instead of several Read operations (b_1, \dots, b_s).

8.4 Regularity

Lemma 7. *Queue, Stack, Register, and Mutex are co-regular.*

Proof. We have a generic schema to build the automaton, which is first to characterize a violation by the existence of a cycle of some kind, and then build an automaton recognizing such cycles. For some of the rules, we prove that these cycles can always be bounded, thanks to a *small model property*. For the others, even though the cycles can be unbounded, we can still build an automaton

(Queue) The empty automaton proves that R_0 and R_{Enq} are regular, as there is no execution e' such that $\text{last}(e') = R$ and $e' \not\sqsubseteq MR$ for $R \in \{R_0, R_{Enq}\}$. The proofs for R_{EnqDeq} and $R_{DeqEmpty}$ are more complicated and can be found respectively in Lemma 16 and Lemma 19

(Stack) The proofs can be found in Appendix 8.7.

(Register and Mutex) Similarly to the rule R_{EnqDeq} , we can reprove Lemma 12 (with sublemmas 13, 14 and 15) to get a small model property, and build an automaton for the small violations.

8.5 Regularity of R_{EnqDeq}

Lemma 12. *Given a history h , if $\forall d_1, d_2 \in \mathbb{D}_h$, $h_{\{d_1, d_2\}} \sqsubseteq R_{EnqDeq}$, then $h \sqsubseteq R_{EnqDeq}$.*

Proof. We first identify constraints which are sufficient to prove that $h \sqsubseteq R_{EnqDeq}$.

Lemma 13. *Let h be a history and x a data value of \mathbb{D}_h . If $Enq(x) \not\prec Deq(x)$, and for all operations o , we have $Enq(x) \not\prec o$, and for all Deq operations o , we have $Deq(x) \not\prec o$, then h is linearizable with respect to MR_{EnqDeq} .*

Proof. We define a graph G whose nodes are the element of h , and whose edges include both the happens-before relation as well as the constraints depicted given by the Lemma. G is acyclic by assumption and any total order compatible with G corresponds to a linearization of h which is in MR_{EnqDeq} .

Given $d_1, d_2 \in \mathbb{D}_h$, we denote by $d_1 \mathbf{W}_{h, MR} d_2$ the fact that $h_{\{d_1, d_2\}}$ is linearizable with respect to R , by using d_1 as a witness for the existentially quantified x variable. We reduce the notation to $d_1 \mathbf{W} d_2$ when the context is not ambiguous.

First, we show that if the same data value can be used as a witness for x for all projections of size 2, then we can linearize the whole history (using this same data value as a witness).

Lemma 14. *For $d_1 \in \mathbb{D}_h$, if $\forall d \neq d_1$, $d_1 \mathbf{W} d$, then $h \sqsubseteq MR_{EnqDeq}$.*

Proof. Since $\forall d \neq d_1$, $d_1 \mathbf{W} d$, the happens-before relation of h respects the constraints given by Lemma 13, and we can conclude that $h \sqsubseteq MR_{EnqDeq}$.

Next, we show the key characterization, which enables us to reduce non-linearizability with respect to MR_{EnqDeq} to the existence of a cycle in the \mathbf{W} relation.

Lemma 15. *If $h \not\sqsubseteq MR_{EnqDeq}$, then h has a cycle $d_1 \mathbf{W} d_2 \mathbf{W} \dots \mathbf{W} d_m \mathbf{W} d_1$.*

Proof. Let $d_1 \in \mathbb{D}_h$. By Lemma 14, we know there exists $d_2 \in \mathbb{D}_h$ such that $d_1 \mathbf{W} d_2$. Likewise, we know there exists $d_3 \in \mathbb{D}_h$ such that $d_2 \mathbf{W} d_3$. We continue this construction until we form a cycle.

We can now prove the small model property. Assume $h \not\sqsubseteq R$. By Lemma 15, it has a cycle $d_1 \mathbf{W} d_2 \mathbf{W} \dots \mathbf{W} d_m \mathbf{W} d_1$. If there exists a data-value x such that $Deq(x)$ happens-before $Enq(x)$, then $h_{\{x\}} \not\sqsubseteq R_{EnqDeq}$, which contradicts our assumptions.

For each i , there are two possible reasons for which $d_i \not\ll d_{(i \bmod m)+1}$. The first one is that $Eng(d_i)$ is not minimal in the subhistory of size 2 (reason (a)). The second one is that Deq_{d_i} is not minimal with respect to the Deq operations (reason (b)).

We label each edge of our cycle by either (a) or (b), depending on which one is true (if both are true, pick arbitrarily). Then, using the interval order property, we have that, if $d_i \not\ll d_{(i \bmod m)+1}$ for reason (a), and $d_j \not\ll d_{(j \bmod m)+1}$ for reason (a) as well, then either $d_i \not\ll d_{(j \bmod m)+1}$, or $d_j \not\ll d_{(i \bmod m)+1}$ (for reason (a)). This enables us to reduce the cycle and leave only one edge for reason (a).

We show the same property for (b). This allows us to reduce the cycle to a cycle of size 2 (one edge for reason (a), one edge for reason (b)). If d_1 and d_2 are the two data-values appearing in the cycle, we have: $h_{\{d_1, d_2\}} \not\sqsubseteq R_{EngDeq}$, which is a contradiction as well.

Lemma 16. *The rule R_{EngDeq} is co-regular.*

Proof. We prove in Lemma 12 that a differentiated history h has a projection h' such that $\text{last}(h') = R_{EngDeq}$ and $h' \not\sqsubseteq MR_{EngDeq}$ if and only if it has such a projection on 1 or 2 data-values. Violations of histories with two values are: *i*) there is a value x such that $Deq(x)$ happens-before $Eng(x)$ (or $Eng(x)$ doesn't exist in the history) or *ii*) there are two operations $Deq(x)$ in h or, *iii*) there are two values x and y such that $Eng(x)$ happens-before $Eng(y)$, and $Deq(y)$ happens-before $Deq(x)$ ($Deq(x)$ doesn't exist in the history).

The automaton $\mathcal{A}_{R_{EngDeq}}$ in Fig. 3 recognizes all such small violations (top branch for *i*, middle branch for *ii*, bottom branch for *iii*).

Let \mathcal{I} be any data-independent implementation. We show that

$$\mathcal{A}_{R_{EngDeq}} \cap \mathcal{I} \neq \emptyset \iff \exists e \in \mathcal{I}, e' \in \text{proj}(e). \text{last}(e') = R_{EngDeq} \wedge e' \not\sqsubseteq MR_{EngDeq}$$

(\Rightarrow) Let $e \in \mathcal{I}$ be an execution which is accepted by $\mathcal{A}_{R_{EngDeq}}$. By data independence, let $e_{\neq} \in \mathcal{I}$ be a renaming such that $e = r(e_{\neq})$, and assume without loss of generality that r doesn't rename the data-values 1 and 2. If e is accepted by the top or middle branch of $\mathcal{A}_{R_{EngDeq}}$, we can project e_{\neq} on value 1 to obtain a projection e' such that $\text{last}(e') = R_{EngDeq}$ and $e' \not\sqsubseteq MR_{EngDeq}$. Likewise, if e is accepted by the bottom branch, we can project e_{\neq} on $\{1, 2\}$, and obtain again a projection e' such that $\text{last}(e') = R_{EngDeq}$ and $e' \not\sqsubseteq MR_{EngDeq}$.

(\Leftarrow) Let $e_{\neq} \in \mathcal{I}_{\neq}$ such that there is a projection e' such that $\text{last}(e') = R_{EngDeq}$ and $e' \not\sqsubseteq MR_{EngDeq}$. As recalled at the beginning of the proof, we know e_{\neq} has to contain a violation of type *i*, *ii*, or *iii*. If it is of type *i* or *ii*, we define the renaming r , which maps x to 1, and all other data-values to 2. The execution $r(e_{\neq})$ can then be recognized by the top or middle branch of $\mathcal{A}_{R_{EngDeq}}$ and belongs to \mathcal{I} by data independence.

Likewise, if it is of type *iii*, r will map x to 1, and y to 2, and all other data-values to 3, so that $r(e_{\neq})$ can be recognized by the bottom branch of $\mathcal{A}_{R_{EngDeq}}$.

8.6 Regularity of $R_{DeqEmpty}$

We first define the notion of *gap*, which intuitively corresponds to a point in an execution where the Queue could be empty.

Definition 9. Let h be a differentiated history and o an operation of h . We say that h has a gap on operation o if there is a partition of the operations of h into $L \uplus R$ satisfying:

- L has no unmatched Enq operation, and
- no operation of R happens-before an operation of L or o , and
- no operation of L happens-after o .

Lemma 17. A differentiated history h has a projection h' such that $\text{last}(h') = R_{\text{DeqEmpty}}$ and $h' \not\sqsubseteq MR_{\text{DeqEmpty}}$ if and only there exists a DeqEmpty operation o in h such that there is no gap on o .

Proof. (\Rightarrow) Assume there exists a projection h' such that $\text{last}(h') = R_{\text{DeqEmpty}}$ and $h' \not\sqsubseteq MR_{\text{DeqEmpty}}$. Let o be a DeqEmpty operation in h' (exists by definition of last).

Assume by contradiction that there is a gap on o . By the properties of the gap, we can linearize h' into a sequential execution $u \cdot o \cdot v$ where u and v respectively contain the L and R operations of the partition.

(\Leftarrow) Assume there exists a DeqEmpty operation o in h such that there is no gap on o . Let h' be the projection which contains all the operations of h as well as o , except the other DeqEmpty operations.

Assume by contradiction that there exists a sequential execution $w \in MR_{\text{DeqEmpty}}$ such that $h' \sqsubseteq w$. By definition of MR_{DeqEmpty} , w can be decomposed into $u \cdot o \cdot v$ such that u has no unmatched operation. Let L be the operations of u , and R the operation of v . Since $h' \sqsubseteq w$, the partition $L \uplus R$ forms a gap on operation o .

We exploit the characterization of Lemma 17 by showing how we can recognize the existence of gaps in the next two lemmas. First, we define the notion of *left-right constraints* of an operation, and show that this constraints have a solution if and only if there is a gap on the operation.

Definition 10. Let h be a distinguished history, and o an operation of h . The left-right constraints of o is the graph G where:

- the nodes are \mathbb{D}_h , the data-values of h , to which we add a node for o ,
- there is an edge from data-value d_1 to o if $\text{Enq}(d_1)$ happens-before o ,
- there is an edge from o to data-value d_1 if o happens-before $\text{Deq}(d_1)$,
- there is an edge from data-value d_1 to d_2 if $\text{Enq}(d_1)$ happens before $\text{Deq}(d_2)$.

Lemma 18. Let h be a differentiated history and o an operation of h . Let G be the graph representing the left-right constraints of o . There is a gap on o if and only if G has no cycle going through o .

Proof. (\Rightarrow) Assume that there is a gap on o , and let $L \uplus R$ be a partition corresponding to the gap. Assume by contradiction there is a cycle $d_m \rightarrow \dots \rightarrow d_1 \rightarrow o \rightarrow d_m$ in G (which goes through o). By definition of G , and since $o \rightarrow d_m$, and by definition of a gap, we know that all operations with data-value d_m must be in R . Since $d_m \rightarrow d_{m-1}$, the operations with data-value d_{m-1} must be in R as well. We iterate this reasoning until we deduce that d_1 must be in R , contradicting the fact that $d_1 \rightarrow o$.

(\Leftarrow) Assume there is no cycle in G going through o . Let L be the set of operations having a data-value d which has a path to o in G , and let R be the set of other operations. By definition of the left-right constraints G , the partition $L \uplus R$ forms a gap for operation o .

Corollary 1. *A differentiated history h has a projection h' such that $\text{last}(h') = R_{\text{DeqEmpty}}$ and $h' \not\sqsubseteq MR_{\text{DeqEmpty}}$ if and only if it has a DeqEmpty operation o and data-values $d_1, \dots, d_m \in \mathbb{D}_h$ such that:*

- *$\text{Enq}(d_1)$ happens-before o in h , and*
- *$\text{Enq}(d_i)$ happens before $\text{Deq}(d_{i-1})$ in h for $i > 1$, and*
- *o happens-before $\text{Deq}(d_m)$, or $\text{Deq}(d_m)$ doesn't exist in h .*

We say that o is covered by d_1, \dots, d_m .

Proof. By definition of the left-right constraints, and following from Lemmas 17 and 18.

Lemma 19. *The rule R_{DeqEmpty} is co-regular.*

Proof. See Section 4.

8.7 Regularity of the Stack rules

Lemma 20. *A differentiated history h has a projection h' such that $\text{last}(h') = R_{\text{PushPop}}$ and $h' \not\sqsubseteq MR_{\text{PushPop}}$ if and only if there exists a projection such that $\text{last}(h') = R_{\text{PushPop}}$ and either*

- *there exists an unmatched $\text{Pop}(d)$ operation in h' , or*
- *there is a $\text{Pop}(d)$ which happens-before $\text{Push}(d)$ in h' , or*
- *for all $\text{Push}(d)$ operations minimal in h' , there is no gap on $\text{Pop}(d)$ in $h' \setminus d$.*

Proof. Similar to Lemma 17.

Lemma 21. *A differentiated history h has a projection h' such that $\text{last}(h') = R_{\text{PushPop}}$ and $h' \not\sqsubseteq MR_{\text{PushPop}}$ if and only if either:*

- *there exists an unmatched $\text{Pop}(d)$ operation, or*
- *there is a $\text{Pop}(d)$ which happens-before $\text{Push}(d)$, or*
- *there exist a data-value $d \in \mathbb{D}_h$ and data-values $d_1, \dots, d_m \in \mathbb{D}_h$ such that*
 - *$\text{Push}(d)$ happens-before $\text{Push}(d_i)$ for every i ,*
 - *$\text{Pop}(d)$ is covered by d_1, \dots, d_m .*

Proof. (\Leftarrow) We have three cases to consider

- *there exists an unmatched $\text{Pop}(d)$ operation: define $h' = h_{\setminus \{d\}}$,*
- *there is a $\text{Pop}(d)$ which happens-before $\text{Push}(d)$: define $h' = h_{\setminus \{d\}}$,*
- *there exist a data-value $d \in \mathbb{D}_h$ and data-values $d_1, \dots, d_m \in \mathbb{D}_h$ such that*
 - *$\text{Push}(d)$ happens-before $\text{Push}(d_i)$ for every i*
 - *$\text{Pop}(d)$ is covered by d_1, \dots, d_m .*

Define $h' = h_{\setminus \{d, d_1, \dots, d_m\}}$. We have $\text{last}(h') = R_{\text{PushPop}}$ because h' doesn't contain PopEmpty operations nor unmatched Push operations. Assume by contradiction that $h' \sqsubseteq MR_{\text{PushPop}}$, and let $w \in MR_{\text{PushPop}}$ such that $h' \sqsubseteq u$. Since $\text{Push}(d)$ happens-before $\text{Push}(d_i)$ (for every i) the witness x of $w \in MR_{\text{PushPop}}$ has to be the data-value d . This means that $w = \text{Push}(d) \cdot u \cdot \text{Pop}(d) \cdot v$ for some u and v with no unmatched Push .

Thus, there is a gap on operation $\text{Pop}(d)$ in $h' \setminus d$, and that $\text{Pop}(d)$ cannot be covered by d_1, \dots, d_m .

(\Rightarrow) Let h' be a projection of h such that $\text{last}(h') = R_{\text{PushPop}}$ and $h' \not\sqsubseteq MR_{\text{PushPop}}$. Assume there are no unmatched $\text{Pop}(d)$ operation, and that for every d , $\text{Pop}(d)$ doesn't happens-before $\text{Push}(d)$. This means that h' is made of pairs of $\text{Push}(d)$ and $\text{Pop}(d)$ operations.

Let $\text{Push}(d)$ be a Push operation which is minimal in h' . We know there is one, because we assumed that $\text{last}(h') = R_{\text{PushPop}}$, and we know that there is a Push which is minimal because for every d , $\text{Pop}(d)$ doesn't happens-before $\text{Push}(d)$.

By Lemma 20, we know that there is no gap on $\text{Pop}(d)$. Similarly to Lemma 18 and Corollary 1, we deduce that there are data-values $d_1, \dots, d_m \in \mathbb{D}_{h'}$ such that $\text{Pop}(d)$ is covered by d_1, \dots, d_m . Our goal is now to prove that we can choose d and d_1, \dots, d_m such that, besides these properties, we also have that $\text{Push}(d)$ happens-before $\text{Push}(d_i)$ for every i . Assume there exists i such that $\text{Push}(d)$ doesn't happen-before $\text{Push}(d_i)$. We have two cases, either $\text{Pop}(d)$ is covered by $d_1, \dots, d_{i-1}, d_{i+1}, \dots, d_m$, in which case we can just get rid of d_i ; or this is not the case, and we can choose our new d to be d_i and remove d_i from the list of data-values. We iterate this until we have a data-value $d \in \mathbb{D}_h$ such that

- $\text{Push}(d)$ happens-before $\text{Push}(d_i)$ for every i ,
- $\text{Pop}(d)$ is covered by d_1, \dots, d_m .

Lemma 22. *The rule R_{PushPop} is co-regular.*

Proof. The automaton Fig. 4 recognizes the violations given by Lemma 21. The proof is then similar to Lemma 19.

Lemma 23. *The rule R_{Push} is co-regular.*

Proof. We can make a characterization of the violations similar to Lemma 21. This rule is in a way simpler, because the Push in this rule plays the role of the Pop in R_{PushPop} .

Lemma 24. *The rule R_{PopEmpty} is co-regular.*

Proof. Identical to Lemma 19 (replace Enq by Push , Deq by Pop , and DeqEmpty by PopEmpty).

8.8 Regular automata used to prove regularity

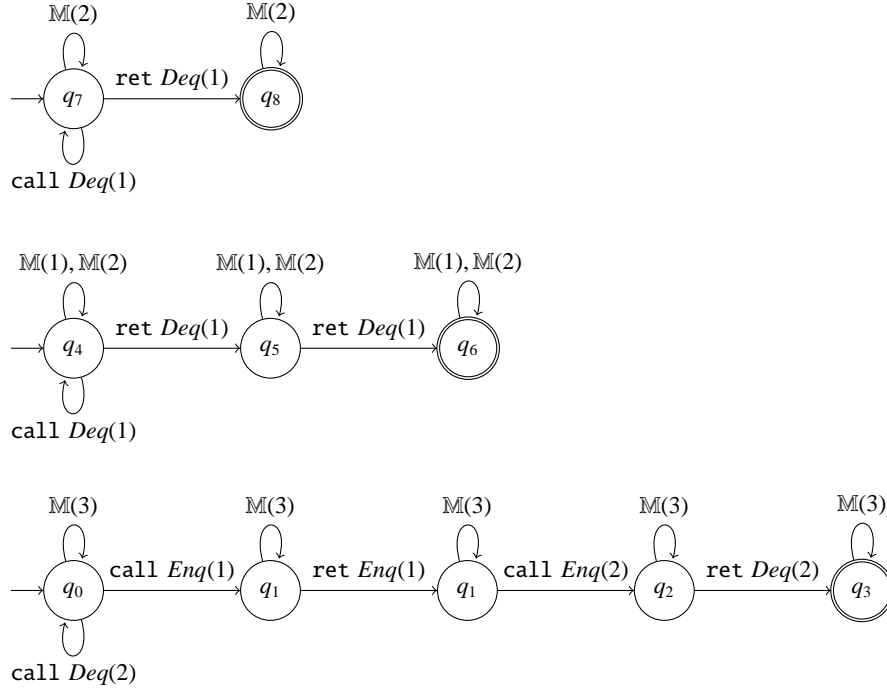


Fig. 3. A non-deterministic automaton recognizing R_{EnqDeq} violations. The top branch recognizes executions which have a *Deq* with no corresponding *Enq*. The middle branch recognizes two *Deq*'s returning the same value, which is not supposed to happen in a differentiated execution. The bottom branch recognizes FIFO violations. By the closure properties of implementations, we can assume the *call Deq(2)* are at the beginning.

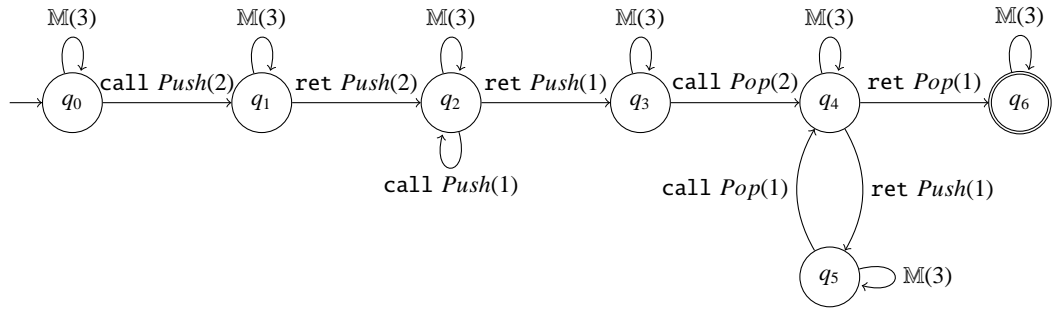


Fig. 4. An automaton recognizing $R_{PushPop}$ violations. Here we have a *Push(2)* operation, whose corresponding *Pop(2)* operation is covered by *Push(1)/Pop(1)* pairs. The *Push(2)* happens before all the pairs. Intuitively, the element 2 cannot be popped from the Stack there is always at least an element 1 above it in the Stack (regardless of how linearize the execution).